Exploring the potential of digital technologies for the teaching and learning of functions: the case of Casyopée

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Abstract
The aim of this paper is to explore the potential of the Casyopée's computer environment for the teaching and learning of functions at upper secondary level. We are particularly interested in the "student" side with a study of appropriate situations using Casyopée and their effects on learning functions. We propose an approach to functions via the functional modelling of geometrical dependencies. The proposed functional modelling cycle helped us to design learning situations, analyze and clarify students' types of activities about functions. Therefore, our study showed how such an approach to functions in a geometrical setting can exist at upper secondary level thanks to the geometric and algebraic environments such as Casyopée.

1. Introduction

Functions play a major role in mathematics and essential in experimental sciences for modelling real world phenomena. The notion of function is central in a wide range of mathematical topics studied especially at the secondary and the upper secondary level [12]. The existing research reports on numerous difficulties related to this notion, especially in coordinating understandings in different settings and dealing with representations in several registers.

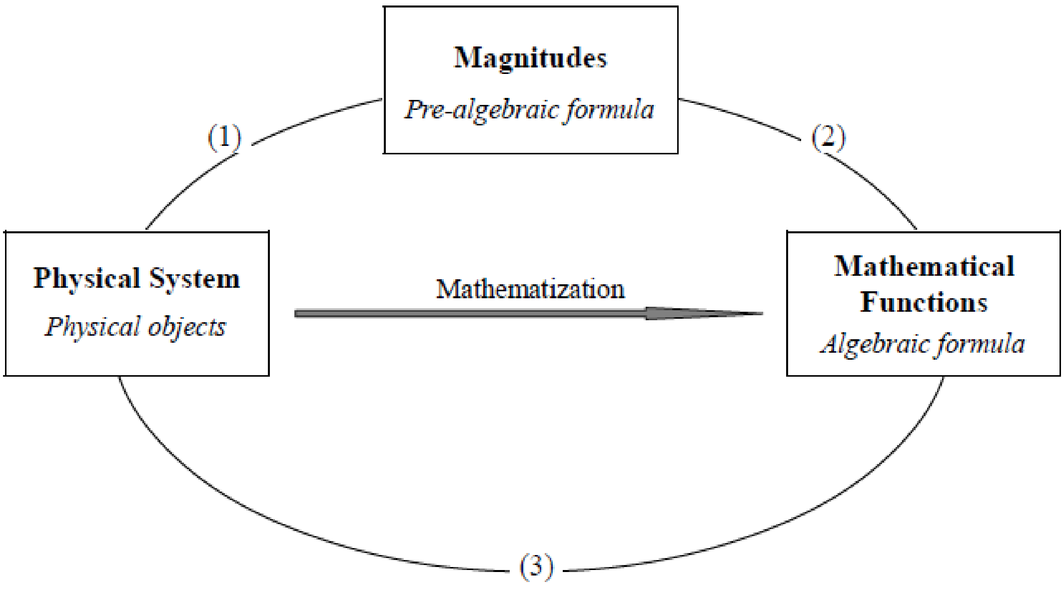
Recently, many authors paid special attention to the potential of digital technologies for the teaching and learning of functions ([1], [9], [7], [10], [3], [12]). Some authors focused on the experience of changes and movements and the understanding of this experience as covariation or dependency relation between magnitudes as key elements leading to the notion of function. For example, having used a motion sensor and a graphing calculator, Arzarello and Robutti [1] asked students to describe different types of movements in terms of mathematical functions by using graphs and tables of values. Then the covariations and dependencies between time and distance in the physical system were directly modelled by mathematical functions. In the Vygotskian perspective of semiotic mediation, Falcade, Laborde and Mariotti [7] chose the Dynamic Geometry (DG) as an application domain to provide students with qualitative experience of covariations and functional dependencies. Their research indicated that the DG tools (Dragging, Trace, Macro…) can be used to help students to understand the concept of function.

Bloch [2] distinguished different settings for the concept of function: numerical, algebraic, geometrical, graphic, formal and analytical. She particularly put the emphasis on the interaction between graphic and formal settings for a didactical engineering focusing on the validation of the properties of function. She said at that time the geometrical setting is rarely used in the teaching and learning of functions. More recently, computer learning environments, especially those offering both geometrical and algebraic capabilities, have been designed with the aim of providing some sort of combination of DG and algebraic multirepresentation. These computer environments could encourage the interplay between two powerful mathematical worlds, different kinds of algebraic expression, as well as new opportunities for direct manipulation of dynamically linked geometric and symbolic forms of mathematical objects [12]. The potential of these computer environments encourage an assumption that digital technologies can contribute to the reintroduction of the geometrical setting in approaching functions.

In this paper we propose an approach that considers functions as *models of dependencies between magnitudes* in a geometrical setting as defined by Bloch [2] and used by Falcade and her colleagues [7]. More specifically, we propose a cycle of functional modeling for approaching functions in computational learning environments. This cycle of functional modelling distinguishes three domains of representation of a functional dependency: Physical System, Magnitudes, and Mathematical Functions. Our general hypothesis is that considering activities at the domain “Magnitudes” allows highlighting functional modelling that links activities about functions in the geometrical setting and in algebra. The activities at this intermediate domain could be fruitful for conceptualizing functions: choosing appropriate variables to quantify observations, distinguishing functional dependencies among covariations, building a pre-algebraic formula expressing the functional dependency… strongly contribute to make functions exist as models of dependencies in Geometry. Our approach to functions takes advantage of the potential of multiple representations offered by computational learning environments.

2. Approaching functions through functional modelling of geometrical dependencies

Many researchers emphasized the covariation view when approaching functions ([16], [5], [4]). The covariation view of function pays special attention to the understanding of the manner in which dependent and independent variables change as well as the coordination between these changes [12]. The essence of a covariation view is the consideration of simultaneous variation of between quantities/magnitudes in different level of dependency. However, according to Lagrange and Psycharis [12], the covariation view of function seems to be not obvious for the students and there is a need for situations that provide students with opportunities to think about the covariational nature of functions in modelling dynamic events.



(1) Creating a calculation, choosing an independent variable and building a pre-algebraic formula expressing the dependency

(2) Expressing algebraically the pre-algebraic formula

(3) Algebraic proof and interpretation of the mathematical solution

Figure 1: *functional modelling cycle*

In a recent work Lagrange and Artigue [10] proposed a typology of activities for function in computational learning environments. Based on this typology, we propose below a functional modelling cycle for approaching functions in computational learning environments. This functional modelling cycle is in the context of solving problems via an algebraic modeling. It doesn't imply that the resolution of a problem is done in a linear way.

In our functional modelling cycle we focus on the intermediate level “Magnitudes” between the “Physical System” (DG in our work) and the “Mathematical Functions”. The process of mathematization is divided into two steps ((1) and (2) in the figure 1). Problems or situations are provided to students in the “Physical System” where they can observe, explore and perceive dependency relations between geometrical magnitudes. At the level “Magnitudes”, students can create a “geometrical calculation” expressing the value of a magnitude, exploring the covariation between magnitudes and measures, choosing a magnitude as an independent variable and another magnitude as a dependent variable then considering the functional dependency (if there exists) between them, building a pre-algebraic formula expressing this functional dependency… These activities are fundamental and fruitful for conceptualizing functions as models of dependency between magnitudes.

The step (2) concerns the process of calculating and representing algebraically the pre-algebraic formula built at the intermediate level “Magnitudes”. At the level “Mathematical Functions” students obtain a mathematical function modelling a functional dependency between quantities/magnitudes in the situation given at the first level “Physical System”. The step (3) includes manipulations, transformations or algebraic proofs to find the mathematical solution of the situation then interpret the mathematical solution.

The “Physical System” (DG in our case) is a level where enactive-iconic explorations are carried out: students can move objects, observe the transformation of the system with the help of technological tools, and start perceiving dependency relations between objects. At the level “Magnitudes” students can use the potential of technological tools to quantify explorations and observations, and make conjectures about the solution of the problem. The construction of a pre-algebraic formula expressing a dependency relation between objects at this level is useful for supporting these observations and explorations. The “Mathematical Functions” is a level where the transformations and algebraic proofs take place. Finally, the return to the physical system aims at interpreting the mathematical solution.

Our approach to functions based on the functional modelling cycle is innovative. Indeed, Falcade and her colleagues [7] for instance focused on the first level: they choose DG as a field to provide students with a qualitative experience of covariation and of functional dependency. Arzarello and Robutti [1] did one of the studies covering the first and third levels, but not the intermediate level “Magnitudes”. The level of quantities/magnitudes (distance, time) was actually taken in charge by the calculator: using implicit variables and units for distance and time, it directly transposed the movement into the mathematical functions with tables and graphs.

We consider the functional modelling cycle as a conceptual framework for the teaching and learning of functions in computational learning environments. It supports our approach where the functions are considered as models of dependencies in an application domain. Indeed, the distinction between three levels of activities helps analyse the progression of functional meanings, highlighting the activities at the intermediate level “Magnitudes” that link sensual experiences and mathematical functions. These activities are fruitful for conceptualizing functions: choosing appropriate variables to quantify observations, distinguishing functional dependencies among covariations, building a pre-algebraic formula expressing the functional dependency… strongly contribute to make functions exist as models of physical dependencies.

3. Casyopée

Casyopée[1] is an open computer environment dedicated to the teaching and learning of functions at upper secondary level. Casyopée has two modules: the symbolic window and the DG window.

The DG window provides the main features of a standard DG system such as the creation and animation of geometric objects. It offers specific aids for modelling a geometrical dependency into an algebraic function thanks to the so-called Geometric Calculation tab. This is a distinguishing feature of Casyopée.

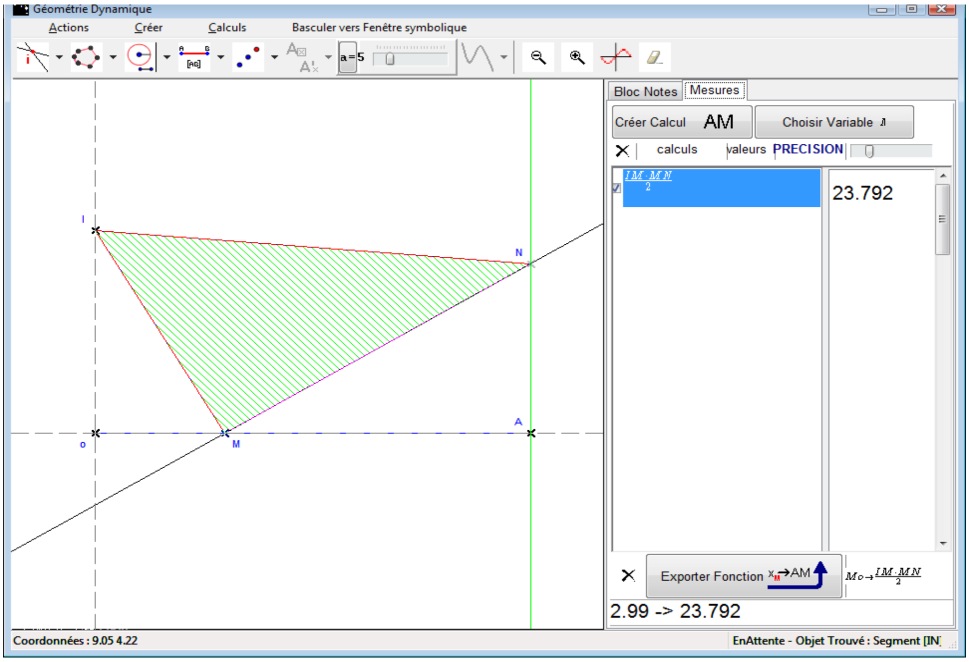


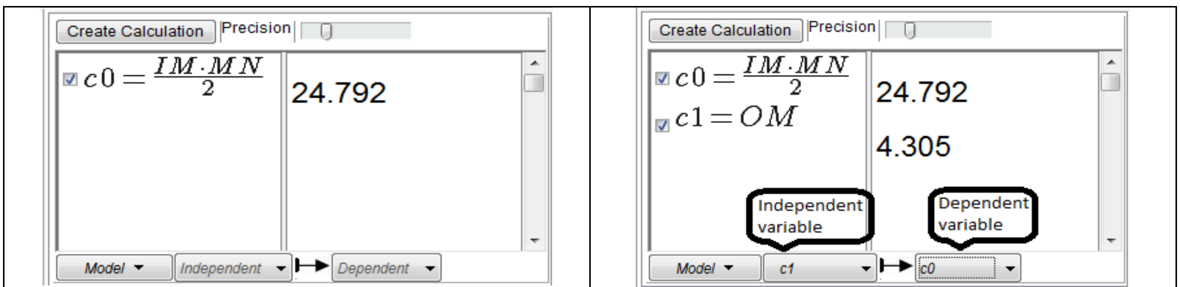
Figure 2: *the Casyopée's Dynamic Geometry window*

Three steps are needed to model algebraically a dependency between two geometrical magnitudes:

(a) creating a "geometrical calculation" expressing the measure of one magnitude which is considered as the dependent variable

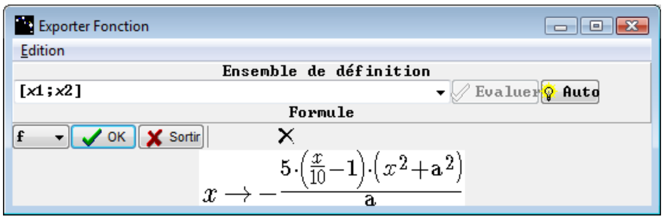
(b) creating a "geometrical calculation" of another magnitude and selecting its measure as an independent variable. For an intended dependent variable, we can choose among several magnitudes an appropriate independent variable to establish a functional dependency

(c) creating an algebraic function modelling this functional dependency between the two selected variables. Casyopée computes automatically the algebraic expression of the function then exports this function into the symbolic window.



a) Creating a dependent variable b) Creating an independent variable OM

[1] A page for downloading Casyopée is available at http://www.casyopee.eu



c) Cassyopée computes automatically the algebraic expression of the function

Figure 3: *Three steps for modelling a geometrical dependency by an algebraic function*

The symbolic window is linked to the geometric window. It provides students with tools to work with functions in the three registers: symbolic, graphic and numeric. It allows defining a function of one independent variable through an algebraic expression and a domain of definition, exploring the graph and displaying the table of values of a function, calculating the derivative of a function and studying its sign… An important feature of Casyopée is that a parameter can have two statuses that a user can switch at any time: animated and formal. In the animated status, a parameter is considered as a placeholder and has a value that the user can change by way of the slider bar. In the formal status, a parameter can be treated formally in algebraic transformations.

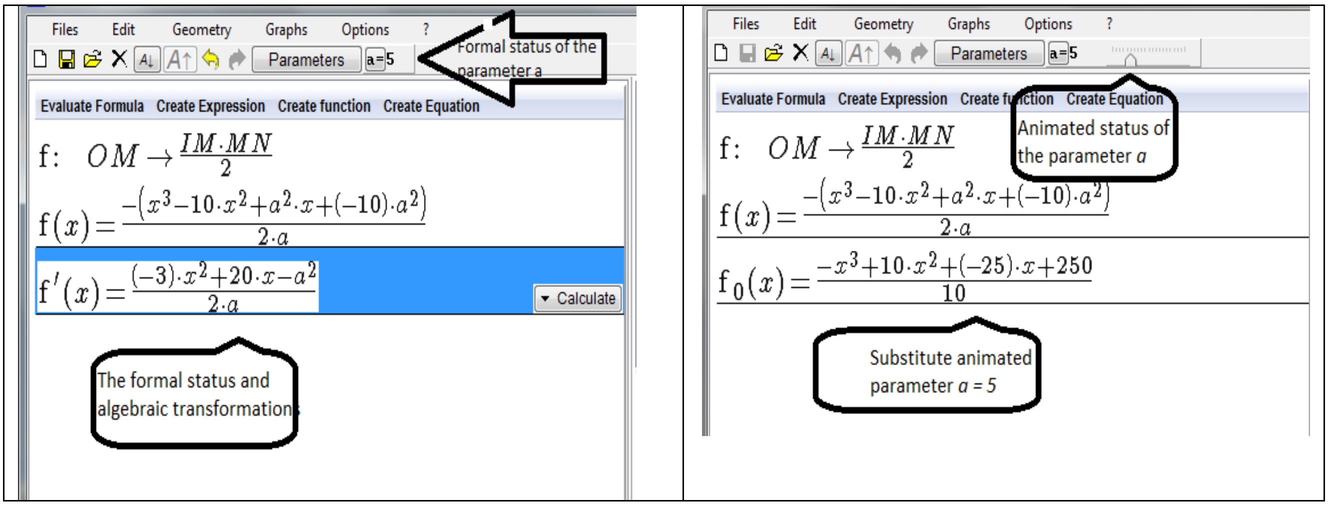


Figure 4: *the animated and formal statuses of a parameter in Casyopée*

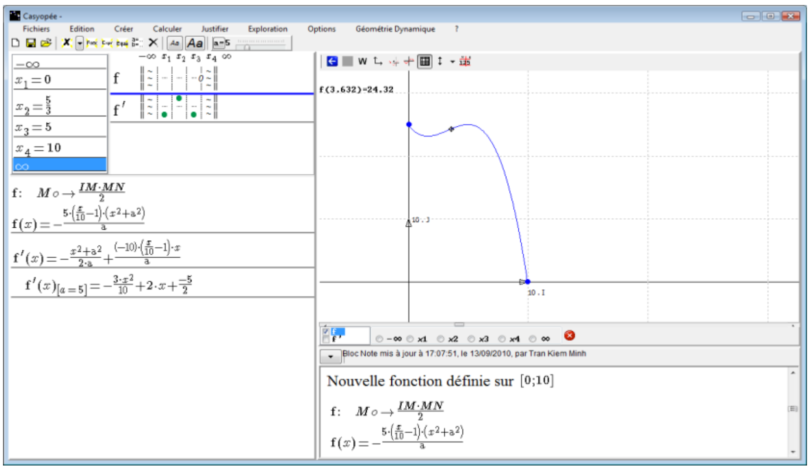


Figure 5: *the Casyopée's symbolic window*

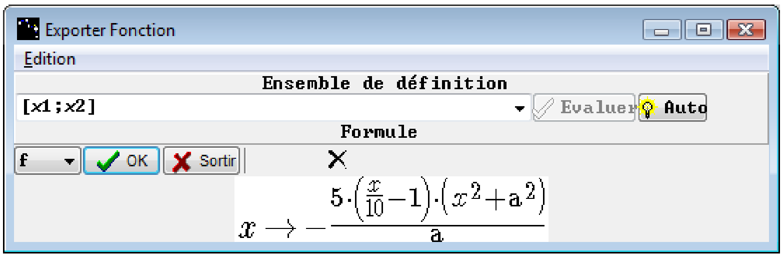


Figure 6: *the exportation form connecting the two windows*

4. Experiments

Consistent with our sensitivity to students' learning with technology, the experiment is organized over a long time of two years and designed in order that students learn about functions while getting acquainted with Casyopée's associated capabilities [11]. The first experiment was designed and implemented in two 11th scientific grades classes in France. It consisted of six sessions and was organized in three parts. The first part focused on capabilities of Casyopée's symbolic window and on quadratic functions. In the second part (two sessions) we first aimed to consolidate students' knowledge of geometrical situations and to introduce them to the geometrical window's capabilities. In the third part students had to solve a problem of maximum area taking advantage of all Casyopée features and of all notions they learnt in the previous sessions.

The second experiment took place in the second year (12th grade). It consisted of three sessions: (1) a session aiming at the consolidation of Casyopée's use some months after the first experiment: the goal is to model a variable area in a square; the function at stake is quadratic, (2) a session where students have to use more completely Casyopée's functionalities, especially for the management of parameters and for symbolic calculation, again in a modelling activity, the function at stake being a third degree parametric polynomial, (3) a session involving the study of a family of logarithm functions, a more classical task with regard to the curriculum as compared with the geometrical modelling in the two other sessions, the goal being that students become aware of how they can use Casyopée to prepare for the baccalaureate, an exam they have to pass at the end of the second year. The semi-directed interview was conducted at the end of the second year, before the baccalaureate in order to understand the evolution students' relationship with mathematics and with Casyopée.

During the two experiments, we observed selected teams of students. In this paper, we focus particularly on two students Elina and Chloé working as a team, which according to the first experiment had a favourable instrumental genesis and were representative of the selected teams. According to their teacher (a member of the Casyopée group), they were good students. The students have been studying Pre-Calculus including the topics such as functions (polynomial functions, rational functions…), limits, derivatives. Students worked in groups with a computer to accomplish a collaborative task.

The type of geometric optimization problem presented in this paper (see below) is previleged by the French new 2009 curriculum for approaching functions with the help of technology. The classroom observations were carried out by way of screen and video recording, and of semi-directed interviews. We consider here a milestone of our classroom observations in the second experiment:

The problem:

In an orthogonal coordinate system *Oxy*, consider the point *I(0; a)*, *a* being a positive parameter and the point *A* of coordinates *(10; 0)*. *M* is a free point on the segment *[OA]*, *N* is on the parallel to the *y*-axis passing by *A* such that the triangle *IMN* is always rectangle in *M*. When *M* is in *A*, then *N* is also in *A*. The problem is to find the position of *M* to maximize the area of the triangle *IMN*.

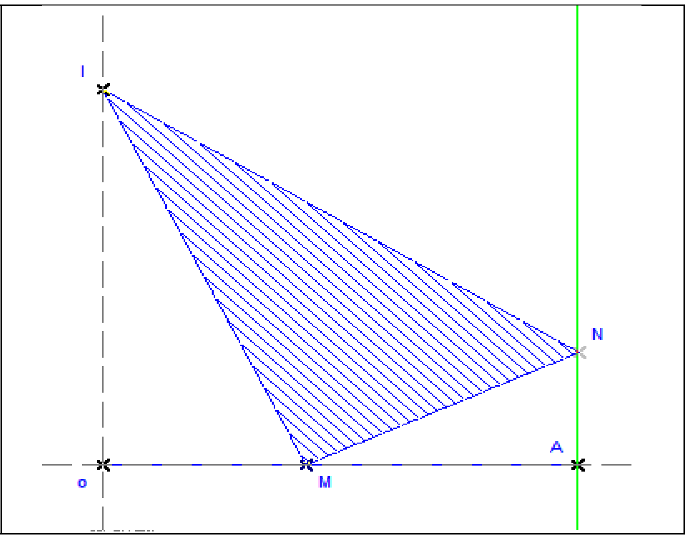


Figure 7: *an optimization problem*

In the worksheet given to the students, they are asked to solve this problem with Casyopée according to the following steps:

- Building a dynamic geometrical figure
- Exploring and conjecturing
- Modelling a geometrical dependency
- Using an algebraic procedure
- Generalising.

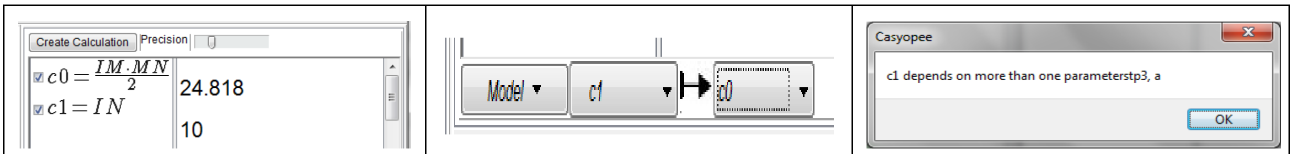
The following table shows the relationship between these mathematical subtasks and the functionalities of the software:

Table 1: *Mathematical subtasks and Casyopée's functionalities*

Mathematical subtasks	**Casyopée's functionalities**
• Building a geometrical figure	• Creating objects in dynamic geometry
• Exploring and conjecturing	• Creating a geometric calculation, dragging free points, observing numeric values
• Modelling a dependency	• Choosing an independent variable, exporting a function
• Using an algebraic procedure	• Using Casyopée's algebraic transformations, and justifications
• Generalising	• Animating parameters

The design of the dynamic figure is made in the Casyopée's DG window. Then students can create a geometrical calculation expressing the area of the triangle, for example $\frac{1}{2}IM \times MN$ to explore the variation of its numerical values. For instance, students can observe the covariation between a magnitude or measure involving the free point M and the area. For the case $a = 5$, when they move the free point M from the origin O to the point A the numerical value of the area decreases then increases until the maximum value of 25 when M is the midpoint of the segment [OA].

After determining a geometrical calculation corresponding to a magnitude whose variation is to study, the functional modelling with Casyopée is characterized by the action to choose a magnitude as an independent variable and then export the resulting functional dependency into the symbolic window. Students can choose an appropriate independent variable among several possible choices: the distance OM, the abscissa of the point M, the distance AM… Casyopée will provide a feedback on the validity of each choice of variable. Note that an inappropriate choice of independent variable implies consequences upon the algebraic expression of function. For example, if students choose the distance MN as an independent variable, this variable is accepted by Casyopée but the algebraic expression of the exported function is too complex. On the other hand, if they take the variable AN Casyopée provide a feedback indicating the dependency is not unique (non functional) and Casyopée cannot compute an algebraic formula expressing this dependency.



a) b) c)

Figure 8 : *(a) Exploring the covariation; (b) considering the functional dependency between the two variables; (c) Casyopée's feedback indicating the inappropriate choice of independent variable c1*

Then students can export the functional dependency into the symbolic window in order to obtain its algebraic model. Casyopée will provide a feedback indicating the domain of definition and an algebraic expression of the function. For instance, with the choice of independent variable *x = OM* , the mathematical function expressing the area of the triangle *IMN* is:

$$f : OM \to \frac{IM.MN}{2}$$

$$f(x) = -\frac{5\left(\frac{x}{10}-1\right)\left(x^2+a^2\right)}{a}$$

By considering the signs of the derivative $f^{'}(x)$, there will be four cases depending on the different values of the parameter *a*:

- $0 < a < 5$: The function $f(x)$ admits two extrema on the interval (0;10) (a local minimum $x_1 = \frac{10-\sqrt{100-3a^2}}{3}$ and a local maximum $x_2 = \frac{10+\sqrt{100-3a^2}}{3}$). In this case, the function $f(x)$ attains its maximal value at $x = x_2 = \frac{10+\sqrt{100-3a^2}}{3}$, that means the area of the triangle *IMN* is maximal for *M* has coordinates $M(\frac{10+\sqrt{100-3a^2}}{3};0)$.
- $a = 5$: The function $f(x)$ admits two extrema on the interval $(0;10)$ (a local minimum $x_1 = \frac{5}{3}$ and a local maximum $x_2 = 5$). In this case, the function $f(x)$ attains its maximal value *25* at $x = x_0 = 0$ or $x = x_2 = 5$, that means the area of the triangle *IMN* is maximal for $M \equiv O$ (O is the origin) or M is the midpoint of the segment $[OA]$.

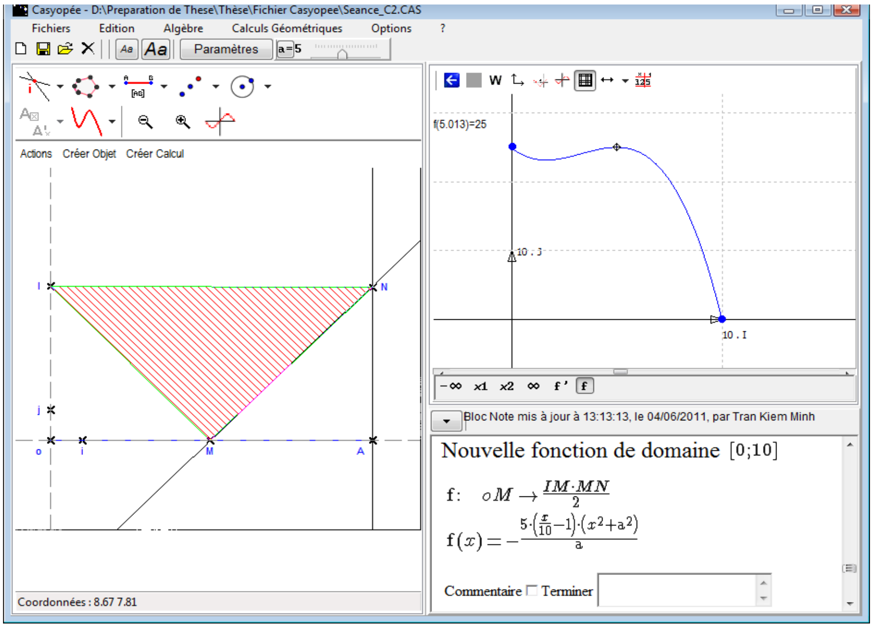
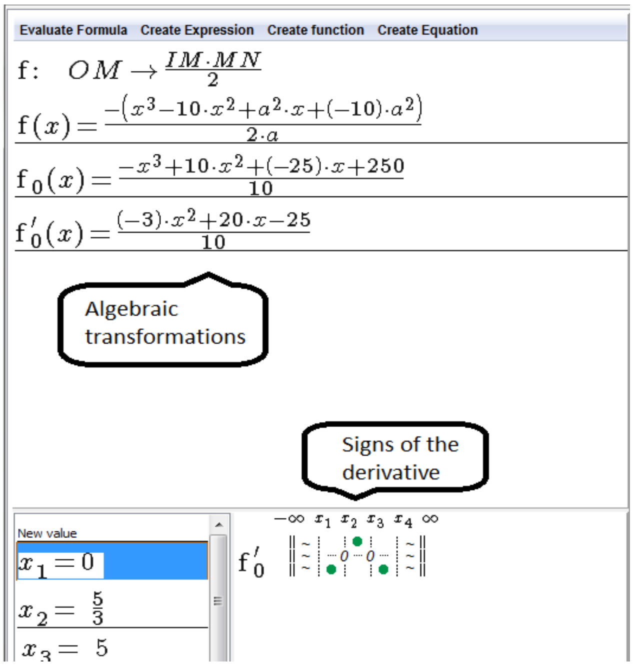


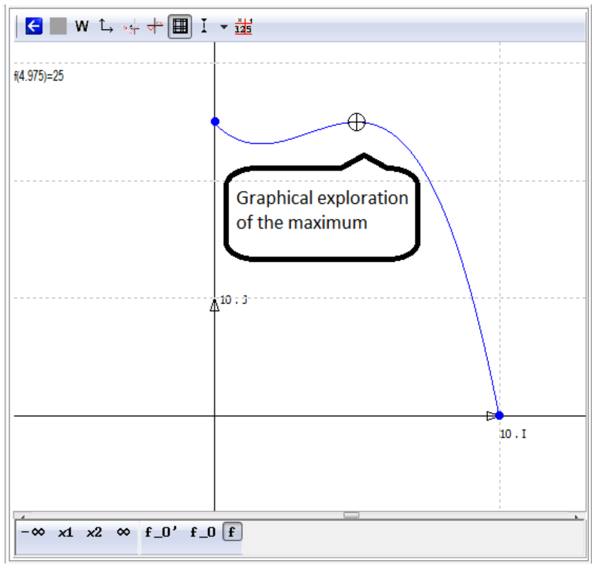
Figure 9: *the dynamic figure of the problem and the graph of the functions for the case* $a = 5$

- $5 < a < \frac{10}{\sqrt{3}}$: The function $f(x)$ admits two extrema on the interval $(0;10)$ (a local minimum $x_1 = \frac{10 - \sqrt{100 - 3a^2}}{3}$ and a local maximum $x_2 = \frac{10 + \sqrt{100 - 3a^2}}{3}$). In this case, the function attains its maximal value *5a* at $x = 0$, that means the area of the triangle IMN is maximal for $M \equiv O$.
- $a \geq \frac{10}{\sqrt{3}}$: The function $f(x)$ is decreasing on the interval $(0;10)$. In this case, the function attains its maximal value *5a* at $x = 0$, that means the area of the triangle IMN is maximal for $M \equiv O$.

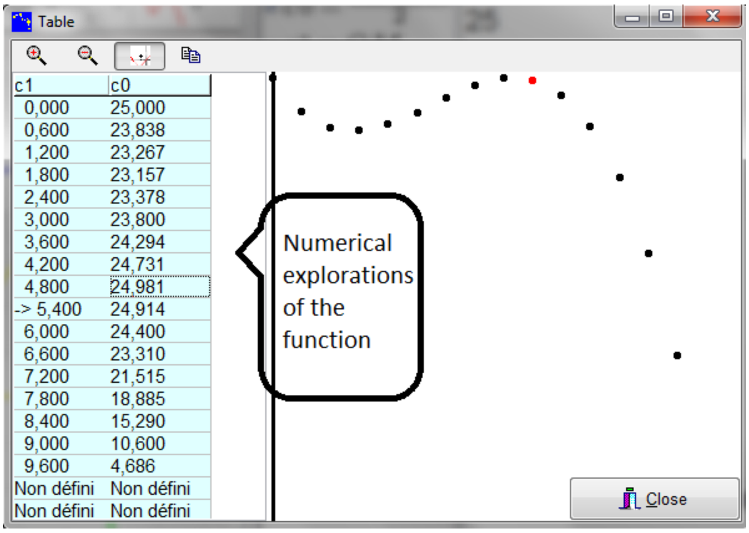
The exportation of a function into the symbolic window corresponds to a change of setting from the geometrical setting to the algebraic setting. In the algebraic setting students can apply different algebraic techniques to the algebraic form of the function and mobilize different semiotic registers [6] in order to find a proof. They can use the graphic register to complete explorations of the variation of the area and the maximum value of the function. They can also pass from the graphic register to the symbolic register to perform algebraic transformations. Finally, a return into the geometric window is necessary for interpreting the mathematical solution of the problem.



a) Algebraic transformations in the symbolic register



b) Explorations in the graphic register



c) Explorations in the numeric register

Figure 10: *Different registers in the algebraic setting*

5. Classroom observations

We interpret here our observations from a point of view of the functional modelling cycle in order to specify students' activities about functions in three levels, together with their difficulties related to these activities.

Passage between the Physical System and the Magnitudes levels:

Creating a geometrical calculation: The observations showed a difficulty in passing from the "Physical System" level to the "Magnitudes" level. The students spent a lot of time to design a geometric figure then create a geometrical calculation expressing the area of the triangle *IMN* and choose an appropriate independent variable.

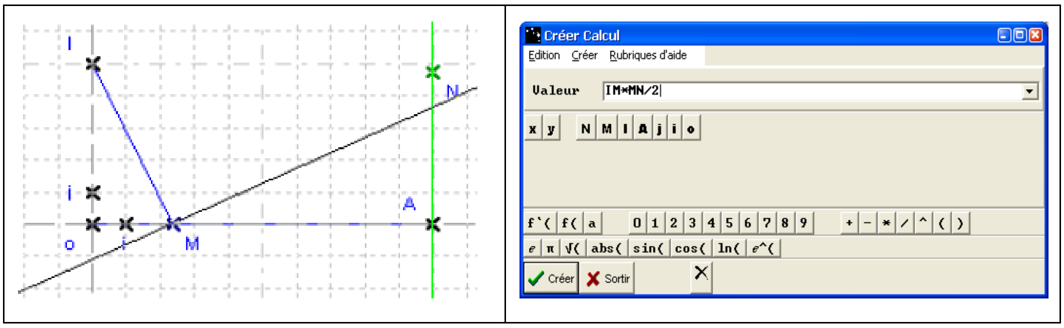
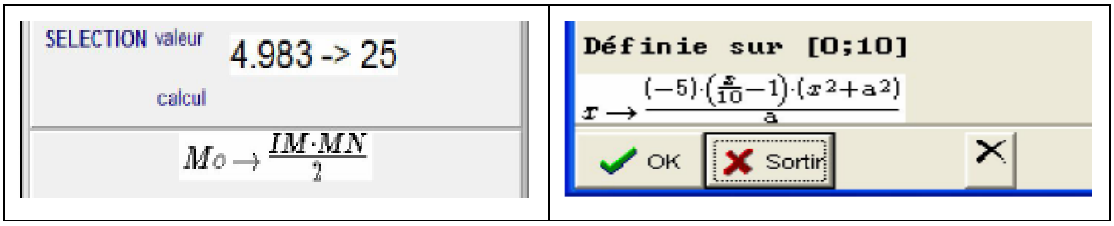


Figure 11: *The passage between the two levels of the team Elina-Chloé*

Choosing an independent variable: The action of choosing an independent variable corresponds to the transition from the type of enactive-iconic activities to the type of generative activities in the same "Magnitudes" level. We found that the choice of an appropriate independent variable is not easy for most of the students. The working in small teams facilitated discussions among themselves to find a good independent variable:

Chloé: Choosing an independent variable? Did we do it with the altitude last time?
Elina: No, the distance OM, I think it will be a good variable.
Chloé: Yes {*She chose this variable then exported the function*}
Elina: Its domain?
Chloé: It is the set of real numbers. Oh no, it is [0;10].
Elina: Look! It is here {*She pointed to the screen*}.
Observer: Finally, what are the steps of the modelling process?
Chloé: One draws a figure and makes conjectures
Elina: One makes a calculation
Chloé: Yes, we draw a figure and make a calculation.
Elina: Then we choose a variable
Chloé: Yes, we export the function and try to validate the conjecture.

Passage between the Magnitudes and the Mathematical Functions levels:
Exporting a function: The students had difficulty in the transition between these two levels, marked by the aids of the teacher. However, the regular use of Casyopée in classroom during the experiment helped them to overtake this difficulty. The feedbacks of Casyopée on the complexity of the algebraic expression of the exported function helped them to find a more appropriate variable.



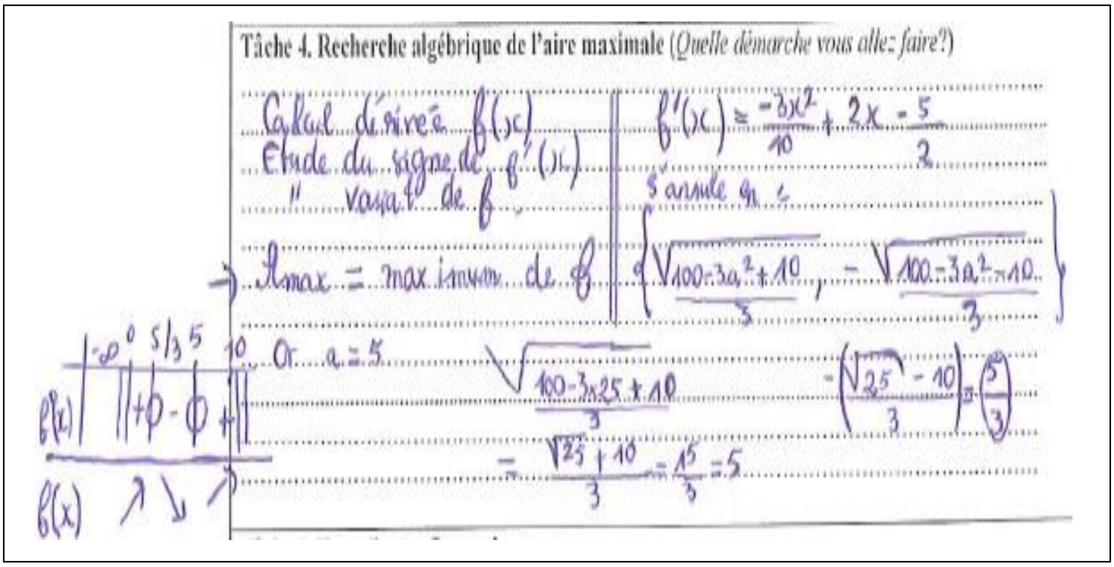


Figure 12: *Passage between these two levels and the algebraic proof of this team*

Passage between the Mathematical Functions and the Physical System levels:
After having found the mathematical solution, the team returned into the DG window of Casyopée to interpret the two possible positions of the point M. The Casyopée's characteristic of integrating the graphics in the DG window has really facilitated the connection between these two levels.

The interview at the end of the experimental sequence:
We report here some of the two students' answers. The first outcome is that after two years of use, the students saw Casyopée as a tool whose appropriation had not been easy:

> *"We did not know all functionalities, tools… in Casyopée. We obtained expressions, but we did not know how to manage them. We did not know which functionalities to use".*

They recognized that these difficulties are linked to the understanding of the mathematical content:

> *"The most difficulty is to choose an independent variable. It is important to choose an appropriate variable".*

In spite of the difficulties observed by the first uses, the students also expressed positive feelings relatively to specificities of Casyopée, especially the help for modelling and the link between symbolic and geometrical windows:

> *"Choosing variables is the interesting part. To perform all the process is great: constructing the figure, table of variation, calculation of the derivative… We have the algebraic and geometrical sides together. We see better how a function "reacts", it is convenient and interesting".*

Students identified clearly different functionalities and how they could help exploring and proving freely:

> *"We can try different variables, animate the figure, and visualize functions (several at the same time), draw a table of signs, find the derivative".*

6. Conclusions

One of the results of our work is to show the possibility of an approach where the functions are considered as models of dependencies in a geometrical setting. In fact, the activities based on the study of dependency relations between magnitudes and measures allowed students to progress in their understanding of the concept of function. Such an approach is consistent with what is currently mentioned by the French curriculum in the teaching and learning of functions at upper secondary level.

Our approach highlighted the functional modelling which links a model in an application domain to a mathematical model. We focused on the "Magnitudes" intermediate level of our functional modelling cycle. From our point of view the students' activities at this level such as creating a geometrical calculation expressing the value of a magnitude, exploring the covariation between magnitudes and measures, choosing an appropriate magnitude as an independent variable to quantify the geometrical functional dependency, computing a mathematical function expressing this functional dependency… are fruitful for conceptualizing functions. Combining this approach with the experimental perspective led us to the construction of appropriate learning situations for approaching functions that have proven effective in the actual context of teaching and learning functions at upper secondary level (Minh, 2012). The functional modelling cycle helped us to design learning situations, analyze and clarify students' types of activities about functions at upper secondary level. Therefore, our study showed how an approach of functions in a geometrical setting can exist at upper secondary level thanks to the geometric and algebraic environments such as Casyopée.

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